Practice Questions for Final Exam - Math 1060Q - Fall 2014

Before anyone asks, the final exam *is* cumulative. It will consist of about 50% problems on exponential and logarithmic functions, 25% problems on trigonometric functions, and 25% problems on other material covered in the course. Most of the problems will look like WebAssign problems, but as always, there will be some problems that require you to know concepts, understand definitions, and/or apply material you've learned in new ways.

You can go over the midterm reviews for more problems from earlier sections. The problems below will give you extra practice on exponential and logarithmic functions. The final exam will include trigonometric identities on it as they were on the second midterm.

You of course should be familiar with the rules of exponents. For one thing, *please* don't mix up $x^{\frac{1}{2}}$ and x^{-2}!

1. Simplify $\sqrt{(x^4x^2)^{\frac{3}{4}}}$.

Solution:

$$\begin{aligned}\sqrt{(x^4x^2)^{\frac{3}{4}}} &= \sqrt{(x^6)^{\frac{3}{4}}}\\ &= \sqrt{x^{\frac{18}{4}}}\\ &= x^{\frac{18}{8}}\\ &= x^{\frac{9}{4}}\end{aligned}$$

2. Simplify $(\frac{3}{x})^{-2}$.

Solution:

$$\begin{aligned}(\frac{3}{x})^{-2} &= (\frac{x}{3})^2\\ &= \frac{x^2}{9}\end{aligned}$$

3. Simplify $\frac{x^{-1}y^2}{y^2x^{-2}}$.

Solution:

$$\begin{aligned}\frac{x^{-1}y^2}{y^2x^{-2}} &= \frac{x^{-1}}{x^{-2}}\\ &= x^{-1}x^2\\ &= x\end{aligned}$$

4. Simplify $(\frac{x^{-2}}{x^{-3}})^{-4}$.

Solution:

$$\begin{aligned}(\frac{x^{-2}}{x^{-3}})^{-4} &= \frac{x^8}{x^{12}}\\ &= \frac{1}{x^4}\end{aligned}$$

5. Simplify $\sqrt{x^{\frac{1}{3}}}x^{\frac{2}{5}}$.

Solution:

$$\begin{aligned}\sqrt{x^{\frac{1}{3}}}x^{\frac{2}{5}} &= (x^{1}3)^{\frac{1}{2}}x^{\frac{2}{5}} \\ &= x^{\frac{1}{6}}x^{\frac{2}{5}} \\ &= x^{\frac{1}{6}+\frac{2}{5}} \\ &= x^{\frac{17}{30}}\end{aligned}$$

6. Show that $(x+y)^2$ is not the same as x^2+y^2.

Solution: Just pick some good numbers to use. If $x = 1$ and $y = 1$, then $(x+y)^2 = (1+1)^2 = 4$, but $x^2 + y^2 = 1^2 + 1^2 = 2$. They're not the same.

7. Simplify $\frac{1}{\frac{1}{x^{-1}}}$.

Solution:

$$\frac{1}{\frac{1}{x^{-1}}} = \frac{1}{x}$$

8. Is $(x+y)^{-1}$ the same as $x^{-1} + y^{-1}$?

Solution: Not at all! You can't distribute exponents. For example, if $x = 1$ and $y = 1$, then $(x+y)^{-1} = \frac{1}{2}$, but $x^{-1} + y^{-1} = 2$.

9. Show that $x^{\frac{1}{2}}$ and x^{-2} are not the same.

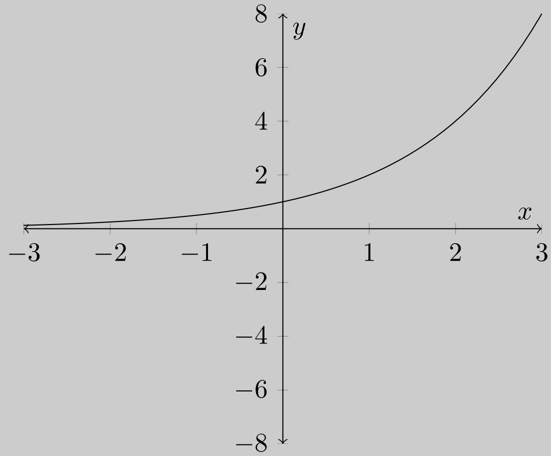
Solution: This is easy to show, but it's a common mistake. $x^{\frac{1}{2}}$ is $\sqrt{x}$, and x^{-2} is $\frac{1}{x^2}$. Just let $x = 4$ and it becomes obvious. (The mistake comes because students will sometimes see the negative sign in x^{-2}, and remember that negative exponents correspond to reciprocals, and so they then take the reciprocal of the exponent...but that's not the case! You have to take the reciprocal of the whole term.)

Now that you understand exponents, you can work on exponential functions. What are their domain? Range? What do their graphs look like?

10. What is the domain of the function $f(x) = 2^x$? What is its range? What does its graph look like? Can you identify 4 points on the graph?

Solution:

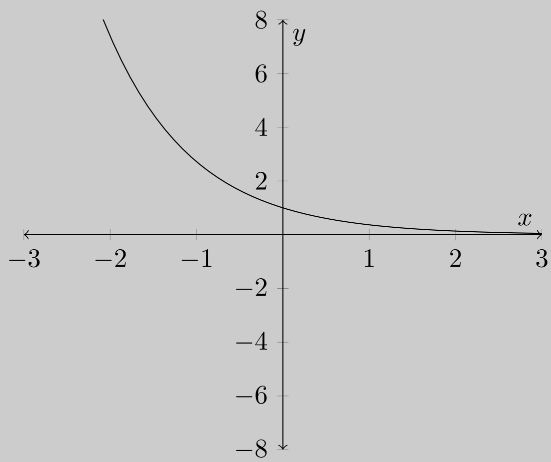
The domain is $(-\infty, \infty)$. The range is $(0, \infty)$. The graph is below. Some points on the graph are $(-1, \frac{1}{2}), (0, 1), (1, 2), (3, 8), (\frac{1}{2}, \sqrt{2})$, etc.



11. What is the domain of the function $g(x) = (\frac{1}{e})^x$? Range? Graph? Identify 4 points?

Solution:

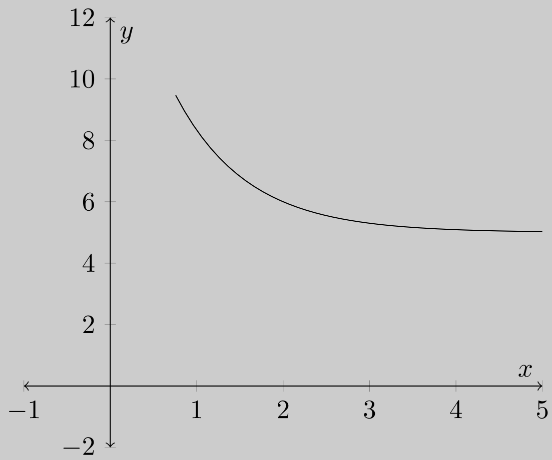
The domain is $(-\infty, \infty)$. The range is $(0, \infty)$. The graph is below. Some points on the graph are $(-1, e), (0, 1), (1, \frac{1}{e}), (3, \frac{1}{e^3}), (\frac{1}{2}, \sqrt{\frac{1}{e}})$, etc.



12. Graph the function $h(x) = (.3)^{x-2} + 5$.

Solution:

Take an exponential graph $(.3)^x$, shift it right by 2, and up by 5.



13. What is the inverse of the function $j(x) = 5e^x$?

Solution: $j^{-1}(x) = \ln(\frac{x}{5})$

14. What is the inverse of the function $k(x) = 2 \cdot 3^{4x+5} + 6$?

Solution: Let $x = 2 \cdot 3^{4y+5} + 6$. Then:

$$\begin{aligned} x - 6 &= 2 \cdot 3^{4y+5} \\ \frac{x-6}{2} &= 3^{4y+5} \\ \log_3(\frac{x-6}{2}) &= 4y + 5 \\ \frac{\log_3(\frac{x-6}{2}) - 5}{4} &= y \end{aligned}$$

So $k^{-1}(x) = \frac{\log_3(\frac{x-6}{2})-5}{4}$.

15. What is the inverse of the function $l(x) = 3\log_4(x-1) + 1$?

Solution:

$$\begin{aligned} x &= 3\log_4(y-1) + 1 \\ \frac{x-1}{3} &= \log_4(y-1) \\ 4^{\frac{x-1}{3}} &= y - 1 \\ 4^{\frac{x-1}{3}} + 1 &= y \end{aligned}$$

Therefore, $l^{-1}(x) = 4^{\frac{x-1}{3}} + 1$.

16. What is the domain of the function $m(x) = \log_2(x+2)$?

Solution: The domain of the function $\log_2(x)$ is $(0, \infty)$. The function $\log_2(x+2)$ is the same as $\log_2(x)$ except shifted 2 to the left, so the domain of $m(x)$ is $(-2, \infty)$.

17. What is the domain of the function $n(x) = \frac{1}{\ln(x-1)}$?

Solution: You can't plug in any negative number to a logarithm function, so that means we have to have $x - 1 > 0$, i.e., $x > 1$. Also, we can't divide by zero, so we have to have $\ln(x-1) \neq 0$. To find when that happens, we solve the equation $\ln(x-1) = 0$ and find that $x = 2$. Putting these together, we find that the domain is $(1, 2) \cup (2, \infty)$.

18. What is the domain of the function $\log_3(x^2 - x)$?

Solution: The domain is all values of x where $x^2 - x > 0$, i.e., where $x^2 > x$, i.e., $(-\infty, 0) \cup (1, \infty)$.

Then we come to logarithms. Can you compute them? (Hint: a few of the below might be undefined.)

19. What is $\ln(e^2)$?

Solution: 2

20. What is $\log_3 \frac{1}{27}$?

Solution: -3

21. What is $\log_2 0$?

Solution: DNE

22. What is $\log_{10} 1$?

Solution: 0

23. What's the approximate value of $\log_2 e$? Is it a number less than 1? Between 1 and 2? Between 2 and 3? Bigger than 3?

Solution: We know $\log_2 2 = 1$, and $\log_2 4 = 2$. Since e is between 1 and 4, $\log_2 e$ must be between 1 and 2. To put it another way, if we're wondering what exponent a will give $2^a = e$, that exponent must be more than 1 and less than 2.

24. What is $\log_5 \sqrt{5}$?

Solution: $\frac{1}{2}$

25. What is $\log_5(-5)$?

Solution: DNE

26. What is $\log_5 \frac{1}{5}$?

Solution: -1

27. What is $\log_5 25$?

Solution: 2

28. What is $\log_5(5^{\frac{3}{2}})$?

Solution: $\frac{3}{2}$

Do you know the rules of logarithms and how to use them to simplify expressions?

29. What is $\log_6(2) + \log_6(3)$?

Solution:
$$\log_6(2) + \log_6(3) = \log_6(2 \cdot 3) = \log_6(6) = 1$$

30. Simplify $\ln 16 - 5\ln 2 + \ln 1$.

Solution:
$$\ln 16 - 5\ln 2 + \ln 1 = \ln 16 - \ln(2^5) + 0 = \ln \frac{16}{2^5} = \ln \frac{1}{2}$$

31. True or false? $(\ln 3)^2 = \ln(3^2)$

Solution: False! $\ln 3$ must be a number a little bit bigger than 1 (since $e^1 = e \approx 2.71$, if we raise the exponent a little bit (to, say, 1.1), then $e^{1.1} \approx 3$...that's very approximate). So $(\ln 3)^2 \approx 1.1^2 = 1.21$. On the other hand $\ln(3^2) = \ln(9) > \ln(e^2) = 2$.

32. True or false? $\frac{\ln x}{\ln y} = \ln x - \ln y$

Solution: False! The rule says $\ln(\frac{x}{y}) = \ln x - \ln y$. Not what this says. Just try it with $x = y = 1$.

33. Simplify $e^{\ln 17} - \log_2(4^{12})$.

Solution:
$$e^{\ln 17} - \log_2(4^{12}) = 17 - 12\log_2(4) = 17 - 12 \cdot 2 = -7$$

34. What is $e^{4\ln\pi}$?

Solution:
$$e^{4\ln\pi} = e^{\ln(\pi^4)} = \pi^4$$

35. What is $\ln(\frac{1}{e^2})^3$?

Solution:
$$\ln(\frac{1}{e^2})^3 = 3\ln(\frac{1}{e^2}) = 3\ln(e^{-2}) = 3(-2) = -6$$

Can you solve equations that involve exponents and logarithms?

36. Solve $\log_7(x+5) - \log_7(x-1) = 2$.

Solution:

$$\begin{aligned}\log_7(x+5)-\log_7(x-1)&=2\\ \log_7(\frac{x+5}{x-1})&=2\\ \frac{x+5}{x-1}&=49\\ x+5&=49x-49\\ 54&=48x\\ x&=\frac{54}{48}\end{aligned}$$

37. Solve $\log_2(x^2)-\log_2(3x-8)=2$.

Solution:

$$\begin{aligned}\log_2(x^2)-\log_2(3x-8)&=2\\ \log_2(\frac{x^2}{3x-8})&=2\\ \frac{x^2}{3x-8}&=4\\ x^2&=12x-32\\ x^2-12x+32&=0\\ (x-8)(x-4)&=0\\ x&=4,8\end{aligned}$$

38. Solve $16^x=45$.

Solution: $x=\log_{16}(45)$

39. Solve $10^{sinx}=1$.

Solution:

40. Solve $2\ln(x+1)-1=0$.

Solution:

$$\begin{aligned}2\ln(x+1)-1&=0\\ \ln(x+1)&=\frac{1}{2}\\ x+1&=\sqrt{e}\\ x&=\sqrt{e}-1\end{aligned}$$

41. Solve $2\ln x=4$.

Solution: $2\ln x=4 \implies \ln x=2 \implies x=e^2$

42. Solve $e^{x^2+2x-3}=1$.

Solution: $e^{x^2+2x-3} = 1 \implies x^2 + 2x - 3 = 0 \implies x = -3, 1$

43. Solve $\log_2(3 - x) = 3$.

Solution:

$$\begin{aligned} \log_2(3 - x) &= 3 \\ 3 - x &= 2^3 \\ -x &= 5 \\ x &= -5 \end{aligned}$$

44. Solve $3^{1-2x} = 27$.

Solution: $3^{1-2x} = 27 \implies 3^{1-2x} = 3^3 \implies 1 - 2x = 3 \implies x = -1$

45. Solve $\log_4(3x) = \frac{1}{2}$.

Solution:

$$\begin{aligned} \log_4(3x) &= \frac{1}{2} \\ 3x &= 4^{\frac{1}{2}} \\ 3x &= 2 \\ x &= \frac{2}{3} \end{aligned}$$

46. Solve $4 \cdot 16^{-3x} = 16^{3x-2}$.

Solution:

$$\begin{aligned} 4 \cdot 16^{-3x} &= 16^{3x-2} \\ \log_{16}(4 \cdot 16^{-3x}) &= \log_{16}(16^{3x-2}) \\ \log_{16}(4) + \log_{16}(16^{-3x}) &= 3x - 2 \\ \frac{1}{2} + (-3x) &= 3x - 2 \\ \frac{5}{2} &= 6x \\ \frac{5}{12} &= x \end{aligned}$$

47. Solve $e^{x-1} - 5 = 5$.

Solution:

$$\begin{aligned} e^{x-1} - 5 &= 5 \\ e^{x-1} &= 10 \\ x - 1 &= \ln 10 \\ x &= \ln 10 + 1 \end{aligned}$$

48. Solve $e^{x+1} = 5^{x+1}$.

Solution:

$$\begin{aligned} e^{x+1} &= 5^{x+1} \\ \ln(e^{x+1}) &= \ln(5^{x+1}) \\ x+1 &= (x+1)\ln(5) \\ x+1 &= x\ln(5)+\ln(5) \\ x - x\ln(5) &= \ln(5)-1 \\ x(1-\ln(5) &= \ln(5)-1 \\ x &= \frac{1-\ln(5)}{\ln(5)-1} \end{aligned}$$

(Optional subsequent explanation: Note that $x = \frac{1-\ln(5)}{\ln(5)-1} = -1$. This makes sense. We know all exponential functions that haven't been shifted/reflected/scaled around cross the y-axis at $(0,1)$, which means that $a^x = b^x$ is true when $x = 0$. Both e^{x+1} and 5^{x+1} have been shifted one unit to the left. So $e^{x+1} = 5^{x+1}$ is true when $x = -1$.)

49. Solve $e^{x+1} = 2 \cdot 3^{x-2}$.

Solution:

$$\begin{aligned} e^{x+1} &= 2\cdot 3^{x-2} \\ x+1 &= \ln(2\cdot 3^{x-2} \\ x+1 &= \ln 2 + (x-2)\ln 3 \\ x+1 &= \ln 2 + x\ln 3 - 2\ln 3 \\ x - x\ln 3 &= \ln 2 - \ln 9 - 1 \\ x(1-\ln 3) &= \ln(\frac{2}{9}) - 1 \\ x &= \frac{\ln(\frac{2}{9})-1}{1-\ln 3} \end{aligned}$$

(Yes, there was some simplification with logs in here that was not strictly necessary. But it's included for you to practice recognizing rules of logarithms.)

50. Solve $2\ln x = \ln(4x+6) - \ln 2$

Solution:

$$\begin{aligned} \ln(x^2) &= \ln(4x+6) - \ln 2 \\ \ln(x^2) &= \ln(\frac{4x+6}{2}) \\ x^2 &= \frac{4x+6}{2} \end{aligned}$$

$$\begin{aligned} 2x^2 - 4x - 6 &= 0 \\ x^2 - 2x - 3 &= 0 \\ (x+1)(x-3) &= 0 \\ x = -1, x &= 3 \end{aligned}$$

(Since x must be in (0,infinity)) x=3

51. Solve $\ln(2x-1)+\ln(3x-2)=\ln 7$

Solution:

$$\begin{aligned} \ln(2x-1)+\ln(3x-2) &= \ln 7 \\ \ln(2x-1)(3x-2) &= \ln 7 \\ (2x-1)(3x-2) &= 7 \\ 6x^2-7x+2 &= 7 \\ 6x^2-7x-5 &= 0 \\ (3x-5)(2x+1) &= 0 \\ x=\frac{5}{3}, x &= \frac{-1}{2} \end{aligned}$$

(Since x must be in (0,infinity)) x=$\frac{5}{3}$

52. Find a solution to $\log_{10}(\tan x)=2$. (Challenge problem.)

Solution: Challenge problem, but it looks harder than it actually is.

$$\begin{aligned} \log_{10}(\tan x) &= 2 \\ \tan x &= 100 \\ x &= \arctan(100) \end{aligned}$$

53. Solve $e^{2x}+(e^x-1)^2=1$.

Solution:

$$\begin{aligned} e^{2x}+e^{2x}+1-2e^x &= 1 \\ 2e^{2x}-2e^x &= 0 \\ e^{2x}-e^x &= 0 \\ e^x(e^x-1) &= 0 \\ e^x-1 &= 0 \\ \ln(e^x) &= \ln(1) \\ x &= 0 \end{aligned}$$

(There is no solution for $e^x=0$ since $e^x>0$ for all x.)

54. Solve $(4\log_{10} x+3)\cdot\log_{10} x^2+1=0$.

Solution:

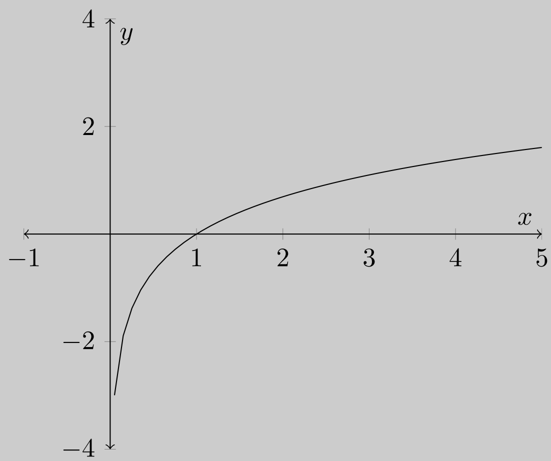
$$\begin{aligned}(4\log_{10}x+3)\cdot 2\log_{10}x+1&=0\\ 8(\log_{10}x)^2+6\log_{10}x+1&=0\\ (2\log_{10}x+1)(4\log_{10}x+1)&=0\\ 2\log_{10}x+1=0 or 4\log_{10}x+1&=0\\ \log_{10}x=-\frac{1}{2} or \log_{10}x&=-\frac{1}{4}\\ x=10^{-\frac{1}{2}} or x&=10^{-\frac{1}{4}}\end{aligned}$$

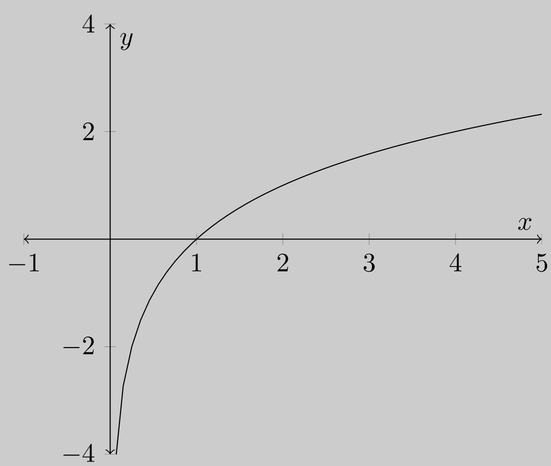
Can you graph logarithmic functions?

55. Graph $\ln x$ and $\log_2 x$.

Solution:

Good exercise: figure out which is which.

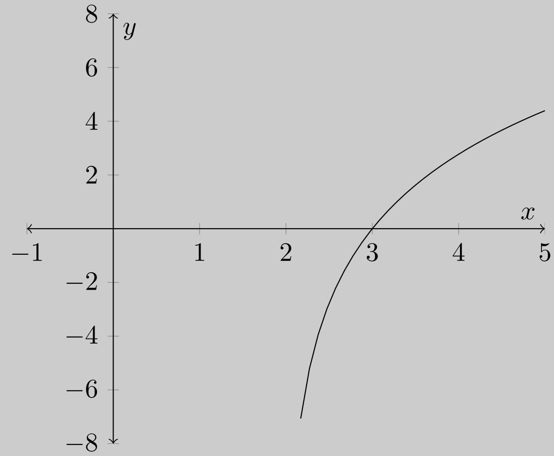




56. What's the relationship between the graphs of $f(x) = \log_3 x$ and $g(x) = 3^x$?

Solution: One is the reflection of the other over the line $y = x$.

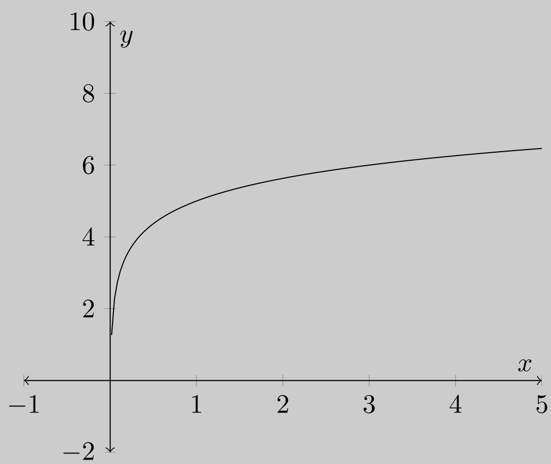
57. Graph $g(x) = 4\ln(x-2)$. Label 4 points on this graph.



Solution:

58. Graph $f(x) = \log_3(x) + 5$. Label 4 points on this graph. Does it have any asymptotes?

Solution: Vertical asymptote at $x = 0$.



Finally, can you use exponential functions to model real-world situations? We have four examples of such situations in this class – population growth, decay of radioactive material, compound interest, and temperature. The first two situations can be modeled using a "non-shifted" exponential equation, $Q(t) = Q_0e^{kt}$. The last one is modeled using a "shifted" exponential equation, $Q(t) = Q_0e^{kt} + C$. Compound interest may be modeled differently, depending on if we are dealing with continuously compounded interest or compounding per period (e.g., monthly, semi-annually). To make things easier on you, we'll only consider continuously compounded interest, which also can be modeled by $Q(t) = Q_0e^{kt}$.

59. Say you invest \$100 into a bank account that earns 10% interest, compounded continuously. How long will it take for you to earn \$300 in interest?

Solution: If we've earned \$300 in interest, that means the account now has \$400 (original

\$100 plus \$300 more). So we need to solve $400 = 100e^{.1t}$ for t.

$$\begin{aligned} 400 &= 100e^{.1t} \\ 4 &= e^{.1t} \\ \ln 4 &= .1t \\ 10 \ln 4 &= t \end{aligned}$$

60. Let's say you have a radioactive isotope with a half life of 100 years. If you start with a sample of 60 mg, how long will it be before there is just 10 mg left?

Solution: If the half life is 100 years, that means if we start with 1 mg, after 100 years we'll have .5 mg. In other words, $.5 = 1 \cdot e^{k \cdot 100}$. We can use this to find k, which helps us express the amount of substance as a function of time. Solving that equation gives $k = \frac{\ln(\frac{1}{2})}{100}$. Therefore, a function for the amount of substance at any given time t if we start with $60mg$ is $Q(t) = 60e^{\frac{\ln(\frac{1}{2})}{100}t}$. We're wondering when there will be 10 mg left; i.e., for what value of t will $Q(t) = 10$? That is, we need to solve the following:

$$\begin{aligned} 10 &= 60e^{\frac{\ln(\frac{1}{2})}{100}t} \\ \frac{1}{6} &= e^{\frac{\ln(\frac{1}{2})}{100}t} \\ \ln(\frac{1}{6}) &= \frac{\ln(\frac{1}{2})}{100}t \\ \frac{\ln(\frac{1}{6})}{\frac{\ln(\frac{1}{2})}{100}} &= t \\ \frac{100\ln(\frac{1}{6})}{\ln(\frac{1}{2})} &= t \end{aligned}$$

Without a calculator it's hard to say what this time is, but you can at least figure out that it should be between 200 and 300 (why?).

61. The population of the United States in 1900 was about 76 million. In the year 2000, it was about 282 million. Assuming that population growth in the United States fits an exponential curve (not necessarily a good assumption), what will the population be in the year 2050?

Solution: Set $Q(t) = Q_0e^{kt}$. Taking our starting time as 1900, we have $Q(0) = 76$ and $Q(100) = 282$.
Now we solve for Q_0 and k. From the first equation we have

$$76 = Q(0) = Q_0e^{k \cdot 0} = Q_0$$

and now plugging in $Q_0 = 76$ into our formula, from the second equation we have

$$Q(100) = 76e^{k \cdot 100} = 282.$$

Thus

$$\begin{aligned} 76e^{100k} &= 282 \\ e^{100k} &= \frac{282}{76} \\ 100k &= \ln\left(\frac{282}{76}\right) \\ k &= \frac{\ln\left(\frac{282}{76}\right)}{100} \end{aligned}$$

So our model is $Q(t) = 76e^{\frac{\ln\left(\frac{282}{76}\right)}{100}t}$.
To find what the population will be in the year 2050, we would like to evaluate $Q(150)$ (since $2050 = 1900 + 150$). We find that the population is:

$$Q(150) = 76e^{\frac{\ln\left(\frac{282}{76}\right)}{100}\cdot 150}$$

62. Take a potato at 20 degrees. Stick it in an oven at 300 degrees. After 10 minutes, the potato is 40 degrees. How long will it be before the potato reaches 200 degrees?

Solution: Set $Q(t) = Q_0e^{kt} + C$. Here, $C = 300$ since our exponential model will have an asymptote at 300 degrees (the potato can't get hotter than 300 degrees from being heated by a 300 degree oven). Now we find two points for our model, $Q(0) = 20$ and $Q(10) = 40$.
Since $Q(0) = 20$ we have that $Q_0e^{k\cdot 0} + 300 = 20$, so $Q_0 = -280$. Now plugging that into our model, and using $Q(10) = 40$ gives us:

$$\begin{aligned} -280e^{10k} + 300 &= 40 \\ e^{10k} &= \frac{-260}{-280} \\ 10k &= \ln\left(\frac{-260}{-280}\right) \\ k &= \frac{\ln\left(\frac{-260}{-280}\right)}{10} \end{aligned}$$

Thus our model is $Q(t) = -280e^{\frac{\ln\left(\frac{-260}{-280}\right)}{10}t} + 300$. To find when the potato will reach 200 degrees, we solve $Q(t) = 200$.

$$\begin{aligned} -280e^{\frac{\ln\left(\frac{-260}{-280}\right)}{10}t} + 300 &= 200 \\ e^{\frac{\ln\left(\frac{-260}{-280}\right)}{10}t} &= \frac{-100}{-280} \\ \frac{\ln\left(\frac{-260}{-280}\right)}{10}t &= \ln\left(\frac{-100}{-280}\right) \\ t &= \frac{10\ln\left(\frac{-100}{-280}\right)}{\ln\left(\frac{-260}{-280}\right)} \end{aligned}$$

63. Say a couple has a child, and plans for that child to attend UConn in 18 years. They approximate that tuition will be around \$200,000 for four years by that time. Let's say they have a bank account that earns 5%, compounded continuously. How much do they have to deposit today in order to have \$200,000 in 18 years?

Solution: The formula for continuously compounded interest is $Q(t) = Q_0e^{kt}$ where Q_0 is the initial deposit and k is the interest rate as a decimal. Since the interest rate is 5%, $k = .05$. Thus we have

$$Q(t) = Q_0e^{.05t}.$$

Since the couple wants \$200,000 in 18 years, we should solve $Q(18) = 200,000$ for Q_0. This gives:

$$\begin{aligned} Q_0 e^{0.5\cdot 18} &= 200,000 \\ Q_0 e^9 &= 200,000 \\ Q_0 &= \frac{200,000}{e^9} \end{aligned}$$

So they should invest $\frac{200,000}{e^9}$ dollars today.

64. A radioactive substance decays by 10% in 20 years. How long will it take to decay by 20%?

Solution: Set $Q(t) = Q_0 e^{kt}$. Since the radioactive substance decays by 10% in 20 years, we have that $Q(20) = .9 \cdot Q(0)$. We can use this equation to solve for k:

$$\begin{aligned} Q_0 e^{20k} &= .9 Q_0 e^{0\cdot k} \\ e^{20k} &= .9 \\ k &= \frac{\ln(.9)}{20} \end{aligned}$$

So our model is $Q(t) = Q_0 e^{\frac{\ln(.9)}{20}t}$. We would like to know how long it takes to decay by 20%, so we would like to solve the following equation for t:

$$\begin{aligned} Q(t) &= .8 \cdot Q(0) \\ Q_0 e^{\frac{\ln(.9)}{20}t} &= .8 \cdot Q(0) \\ e^{\frac{\ln(.9)}{20}t} &= .8 \\ \frac{\ln(.9)}{20}t &= \ln(.8) \\ t &= \frac{20\ln(.8)}{\ln(.9)} \end{aligned}$$

65. Bacteria are growing. Eww, gross. Anyway, there are 1000 bacteria when they start, and they triple in population every 5 hours. How long until there are 10,000 bacteria?

Solution: Set $Q(t) = Q_0 e^{kt}$. $Q_0 = Q(0) = 1000$, so $Q(t) = 1000e^{kt}$.

Since $Q(5) = 3000$, $3000 = 1000e^{5k}$, which solves to $k = \frac{1}{5}\ln(3)$. Therefore, $Q(t) = 1000e^{\frac{1}{5}\ln(3)t}$.

$Q(t) = 10000$ when $10000 = 1000e^{\frac{1}{5}\ln(3)t}$, which solves to $t = \frac{5\ln(10)}{\ln(3)}$ hours.

66. Take a turkey at 180 degrees. Pull it out of the oven, into a room that is 70 degrees. After one hour, the turkey has cooled to 160 degrees. How long until it reaches 150 degrees?

Solution: Set $Q(t) = Q_0 e^{kt} + C$. Here $C = 70$, and $Q(0) = Q_0 + 70 = 180$, so $Q_0 = 110$ and $Q(t) = 110e^{kt} + 70$.

Since $Q(1) = 160$, $160 = 110e^k + 70$, which solves to $k = \ln(9/11)$. Therefore, $Q(t) = 110e^{\ln(9/11)t} + 70$.

$Q(t) = 150$ when $150 = 110e^{\ln(9/11)t} + 70$, which solves to $t = \frac{\ln(8/11)}{\ln(9/11)}$ hours. (Note: that's *not* $\frac{\ln(8)}{\ln(9)}$.)

67. A 50 mg sample of radioactive substance decayed to 10 mg in five years. What is the the half-life of the isotope?

Solution: Set $Q(t) = Q_0 e^{kt}$. $Q_0 = Q(0) = 50$, so $Q(t) = 50e^{kt}$.

Since $Q(5) = 10$, $10 = 50e^{5k}$, which solves to $k = \frac{1}{5}\ln(1/5)$. Therefore, $Q(t) = 50e^{\frac{1}{5}\ln(1/5)t}$.

The half-life is the t with $Q(t) = \frac{1}{2}Q_0 = 25$, so $25 = 50e^{\frac{1}{5}\ln(1/5)t}$, which solves to $t = \frac{5\ln(1/2)}{\ln(1/5)}$ years.

68. A sheet of paper is approximately 0.1 mm thick. By folding it in half, you double its thickness. How many times would you need to fold the sheet of paper in half, so that it would be thick enough to reach the Moon? The distance from the Earth to the Moon is approximately 384,400,000,000 mm.

Solution: Let $Q(t)$ be the thickness (in mm) after folding the sheet in half t times. As before, $Q(t) = Q_0 e^{kt}$, and $Q_0 = Q(0) = 0.1$ (the thickness after folding it 0 times), so $Q(t) = 0.1e^{kt}$.

After folding once, $Q(1) = 2(0.1) = 0.2$, so $0.2 = 0.1e^{k}$, which solves to $k = \ln 2$, so $Q(t) = 0.1e^{\ln(2)t}$.

$Q(t) = 384400000000$ when $384400000000 = 0.1e^{\ln(2)t}$, which solves to $t = \frac{\ln(3844000000000)}{\ln(2)}$ folds.

Note: By the exponent rules, $Q(t)$ simplifies to $0.1(2^t)$, so we could also set $384400000000 = 0.1(2^t)$ and write the solution as $t = \log_2(3844000000000) \approx 41.806$ (which is surprisingly small!).